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A GUIDE TO THE USAGE OF POWER SPECTRUM CALCULATIONS ACCORDING T--ETC(U)
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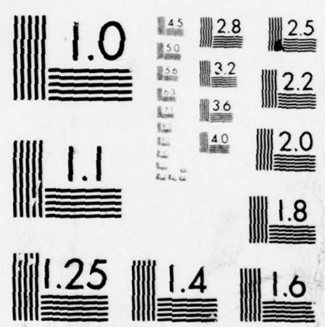

MICROCOPY RESOLUTION TEST CHART
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Columbia University
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R. A. Frosch
Director

Technical Memorandum-to-File No. 47

6 A GUIDE TO THE USAGE OF POWER SPECTRUM CALCULATIONS ACCORDING TO THE METHODS OF TUKEY.

by

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10 T. Arase

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INTRODUCTION

Formulae for the calculation of the estimated power spectrum and its variance from the true value for a stationary, random process given a finite record, have been given by Tukey. ~~(1-3)~~ These methods have been applied by the author to determine the power spectra of temperature records. Since the IBM 650 computation programs ~~(written by A. Anastasio)~~ for the evaluation of these quantities may be generally useful, they will be described in detail. The mathematical background will be developed, with the notation and terminology of Reference 3, and some of the different notations encountered in the extensive literature in this subject will be pointed out.

THEORY

We consider a function

$$G_T(t) = G(t) \qquad -T < t < T$$
$$= 0 \qquad \text{outside.} \tag{1}$$

The Fourier transform of $G_T(t)$ is

$$S(f) = \int_{-\infty}^{\infty} G_T(t) e^{-i\omega t} dt \qquad \omega = 2\pi f \tag{2}$$

and

$$G_T(t) = \int_{-\infty}^{\infty} S(f) e^{i\omega t} df \tag{3}$$

Parseval's Theorem states that

$$\int_{-\infty}^{\infty} |G_T(t)|^2 dt = \int_{-T}^{T} |G(t)|^2 dt = \int_{-\infty}^{\infty} |S(f)|^2 df \tag{4}$$

so that the time average of $G^2(t)$ is

$$\widetilde{G^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(t)|^2 dt \tag{5}$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} |S(f)|^2 df$$

$$= \int_{-\infty}^{\infty} P(f) df$$

where $P(f) = \lim_{T\to\infty} \frac{1}{2T} |S(f)|^2$ (6)

$P(f)df$ represents the contribution to the time average of $G^2(t)$ from frequencies between f and $f+df$ and is called the <u>power spectrum</u> or <u>spectral density</u> or <u>power spectral density</u>. Note that $P(f)$ so defined is a two-sided function of frequency whereas common practice is to use the one-sided power spectrum, $Q(f)$. That is,

$$Q(f) = 2P(f) \tag{7}$$

which is the <u>power density</u> for <u>positive</u> frequencies. The utility of using the two-sided function lies in the simplification of certain integrals.

We define the <u>autocovariance function</u> (also called the autocorrelation function in the literature) to be

$$C(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} G(t)G(t+\tau)dt \tag{8}$$

where τ is the time lag. The variance is $C(0)$. The Fourier transform of $C(\tau)$ is the power spectrum.

$$P(f) = \int_{-\infty}^{\infty} C(\tau)e^{-i\omega\tau}d\tau \tag{9}$$

which follows if one substitutes the Fourier transform of $G(t)$ in the integral (8) and conversely,

$$C(\tau) = \int_{-\infty}^{\infty} P(f)e^{i\omega\tau}df \quad , \tag{10}$$

Since $C(\tau)$ is an even function of τ , the power spectrum is also

$$P(f) = 2\int_{0}^{\infty} C(\tau) \cos \omega\tau d\tau \ . \tag{11}$$

The one-sided power spectrum is

$$Q(f) = 4\int_{0}^{\infty} C(\tau) \cos \omega\tau d\tau \tag{12}$$

which is a form quite often encountered in the literature.

In practice, our records are rather short and we would like to do the processing digitally thereby avoiding the necessity of constructing apparatus for the evaluation of the integrals (8) and (9) and the larger errors inherent in analog processing.

For numerical processing, the data should be read at equal time intervals. Let the $N+1$ numbers

$$Y_o, Y_1, Y_2, \ldots Y_N \tag{13}$$

be the time series, taken at time intervals Δt , with mean

$$\overline{Y} = \frac{1}{N+1}\sum_{q=0}^{N} Y_q \ . \tag{14}$$

Let

$$X_q = Y_q - \overline{Y} \tag{15}$$

be the fluctuation from the mean. We evaluate the integral (8) as the <u>mean lagged product</u>, with the maximum lag m

$$C_r = \frac{1}{N-r}\sum_{q=0}^{N-r} X_q X_{q+r} \qquad r = 0,1,2, \ldots m . \tag{16}$$

Then, the Fourier transform (corresponding to Eq. (11) is the <u>raw spectral density estimate</u>

$$V_r = \Delta t \left[C_o + 2 \sum_{q=1}^{m-1} C_q \cos \frac{qr\pi}{m} + C_m \cos r\pi \right] \tag{17}$$

where V_r is the estimate of the two-sided spectral density centered about the frequency

$$f_r = (r/2m\Delta t) \qquad r = 0,1,2, \ldots m . \tag{18}$$

Factoring the variance, C_o, we find

$$V_r = C_o \Delta t V_r{}' \tag{19}$$

where

$$V_r{}' = 1 + 2 \sum_{q=1}^{m-1} C_q{}' \cos \frac{qr\pi}{m} + C_m{}' \cos r\pi \tag{20}$$

and

$$C_q{}' = (C_q/C_o) .$$

The one-sided power spectrum estimate, W_r is

$$W_r = 2V_r . \tag{21}$$

We note that the sum

$$\sum_{r=0}^{m} W_r \Delta f_r = 2\Delta f \sum_r V_r \tag{22}$$

$$= 2(2m\Delta t)^{-1}(C_0 \Delta tm) \qquad \text{since} \quad \Delta f = \begin{cases} (2m\Delta t)^{-1} & \text{for } r \neq 0,m \\ (4m\Delta t)^{-1} & \text{for } r = 0,m \end{cases}$$

$$= C_0$$

for the contribution from the cosine terms in the summation over r is zero.

An alternate formulation is to refer the power spectrum estimate to unit frequency bands, i.e., $\Delta f = 1$. That is, let

$$L_r = V_r(1/2m\Delta t) = (C_0/2m)V_r' , \tag{23}$$

then,

$$\sum_r 2L_r = 2(1/2m\Delta t)\sum_r V_r = C_0$$

as before.

The V_r are the Fourier transform of the mean lagged products so that formally, the problem is solved. However, the V_r are subject to the errors arising from the mathematical approximation in their evaluation as well as to the complex propagation of errors of the original data. To make a direct computation of these errors involves considerable effort; at least as great as for the computation of the V.

We consider instead the variability to be expected in the power spectrum of a stationary, random Gaussian time series. For example, consider a large number k of oscillators, each with a different frequency, with random phases and with amplitudes in a Maxwellian distribution. The energy radiated per oscillator is $(a^2 + b^2)$ where the amplitude of the radiation is given by $(a \cos \omega t + b \sin \omega t)$. The sum of the squares of the $2k$

numbers, $\sum_k \left(a_k^2 + b_k^2\right)$ is the rate at which the ensemble energy is radiated. This sum has a Chi-squared distribution with 2k degrees of freedom. Hence, if the power spectrum density in any given frequency band depends on ν degrees of freedom, we can use tables of Chi-squared to determine the variability of the calculated power spectrum from the true value. While not strictly correct, this result gives a good estimate of the variability of the power spectrum. The number of degrees of freedom associated with the V_r is $\nu = \frac{2N}{m} - \frac{1}{2}$.

TABLE I

Reliability of power spectrum estimates[4]

Degree of freedom	Possible error			
	2.5%	5%	95%	97.5%
1	1000	250	0.26	0.2
2	40	20	0.33	0.21
5	6	4.37	0.46	0.39
8	3.8	2.8	0.51	0.46
10	3.1	2.6	0.55	0.49
15	2.4	2.1	0.60	0.55
20	2.1	1.8	0.63	0.59
50	1.55	1.45	0.74	0.69
100	1.35	1.29	0.79	0.78

The true value of the spectrum level will lie between 2.6 and 0.55 of the calculated value of 90 percent of the time if there are 10 degrees of freedom for the estimate.

We can determine the effective width, or window, in the frequency domain of the raw spectral density estimates by computing the response to a sine wave.

Let the input signal be

$$X = a \cos \left(2\pi f_o t + \theta\right)$$

which, sampled at time intervals Δt . becomes

$$X_q = a \cos \left(\frac{2\pi q}{P} + \theta\right) \qquad q = 0, 1, 2, \ldots$$

when

$$t = q\Delta t$$

$$f_o = \frac{1}{P\Delta t}$$

then

$$C(r) = \frac{1}{N-r} \sum_{q=0}^{N-r} X_q X_{q+r}$$

$$= \underbrace{\frac{a^2}{2} \cos \frac{2\pi r}{P}}_{\frac{a^2(N-r+1)}{2(N-r)} \cos \frac{2\pi r}{P}} + \frac{a^2}{2} \frac{1}{N-r} \sum_{q=0}^{N-r} \cos \left[\frac{4\pi}{P}\left(q + \frac{r}{2}\right) + 2\theta\right] .$$

The summation becomes

$$\frac{\sin \frac{2\pi}{P}(N-r+1)}{\sin \frac{2\pi}{P}} \cos \left(\frac{2\pi N}{P} + 2\theta\right) \leq 1$$

$\cos\left[(r-N)\frac{2\pi}{P} + 2\theta\right]$

so that we can ignore it compared to the first term. The raw spectral density estimate (Eq. 17) is

$$V_r = 2\Delta t \sum_{q=0}^{m} C_q \cos q \frac{r\pi}{m} - \Delta t \left[C_o + C_m \cos r\pi\right] .$$

M-1   q=1

The first term becomes

$$\Delta t \left(\frac{a^2}{2}\right) \sum_{q=0}^{m} \left[\cos \frac{\pi q u_+}{m} + \cos \frac{\pi q u_-}{m}\right]$$

where

$$u_+ = r + 2m$$

$$u_- = r - 2m$$

which reduces to

$$\frac{\cos\left(\frac{\pi u_+}{2}\right)\sin\left[\frac{\pi u_+}{2}\left(1+\frac{1}{m}\right)\right]}{\sin\frac{\pi u_+}{2m}} + \frac{\cos\left(\frac{\pi u_-}{2}\right)\sin\left[\frac{\pi u_-}{2}\left(1+\frac{1}{m}\right)\right]}{\sin\frac{\pi u_-}{2m}} \quad .$$

For m large, $1+\frac{1}{m} \approx 1$, so

$$V_r = \Delta t\left(\frac{a^2}{2}\right)\left\{\frac{\sin \pi u_+}{2\sin\frac{\pi u_+}{2m}} + \frac{\sin \pi u_-}{2\sin\frac{\pi u_-}{2m}}\right\} - \Delta t\left[C_o + C_m \cos r\pi\right] .$$

In the neighborhood of small u, that is, near the frequency of the sine wave,

$$V_r \approx \frac{\sin \pi u}{2\sin\frac{\pi u}{2m}} \approx m\left(\frac{\sin \pi u}{\pi u}\right)$$

which is 1 for $\pi u = 0$, and has a full-width at half-maximum of

$$\Delta r = \frac{1.9}{\pi} \pm \frac{2m}{P}$$

and has many minor lobes, the largest occurring at

$$r = \frac{3}{4\pi} \pm \frac{2m}{P} \quad ,$$

of amplitude $\left(-\frac{2}{3\pi} \approx -.2\right)$. This is 20 percent of the main peak and is much too large.

By multiplying this filter function by a factor

$$\frac{1}{2}\left(1 + \cos\frac{\pi q}{m}\right)$$

we can reduce the side lobe amplitude to approximately 2 percent of the main peak. The response to a sine wave is now given by

$$U_r = \frac{\Delta t}{2}\left(\frac{a^2}{2}\right)\left\{\frac{\cos\left(\frac{\pi u_+}{2}\right)\sin\left[\frac{\pi u_+}{2}\left(1+\frac{1}{m}\right)\right]}{\sin\frac{\pi u_+}{2m}} + \frac{\cos\left[\frac{\pi}{2}\left(u_+ + 1\right)\right]\sin\left[\frac{\pi}{2}\left(u_+ + 1\right)\left(1+\frac{1}{m}\right)\right]}{2\sin\left[\frac{\pi}{2m}\left(u_+ + 1\right)\right]}\right.$$

$$\left. + \frac{\cos\left[\frac{\pi}{2}\left(u_+ - 1\right)\right]\sin\left[\frac{\pi}{2}\left(u_+ - 1\right)\left(1+\frac{1}{m}\right)\right]}{2\sin\left[\frac{\pi}{2m}\left(u_+ - 1\right)\right]} + \ldots \text{ similar terms for } u_-\right\}$$

which may also be written as

$$U_r = \frac{1}{4}V_{r-1} + \frac{1}{2}V_r + \frac{1}{4}V_{r-1} \quad . \tag{24}$$

These are called the <u>refined spectral density estimates</u> where U_r is the average power density. One can compute that this average is taken over a filter width of $\left(\frac{1}{m\Delta t}\right)$ centered at the frequency f_r .

ERRORS

The finite approximations involved in this computation introduce a number of sources of errors if proper precautions are not taken. Two of the most serious will be mentioned here.

(a) <u>Aliasing</u> - If the sampling interval is Δt , then the highest frequency at which spectral density estimates can be made is $f_N = \frac{1}{2\Delta t}$. If appreciable power is present at frequencies greater than f_N , they will appear at the frequencies

$$f,\ 2f_N \pm f\ ,\ 4f_N \pm f\ ,\ \ldots$$

For example, Fig. 1 shows how the aliased spectrum of a f^{-2} form will appear. Hence, in the design of the experiment, one must insure that the upper frequency cutoff of the measuring instruments is less than f_N. Alternately, low-pass filtering must be resorted to after the data is taken and before it is spectrum analyzed.

(b) Prewhitening - If the true spectrum has large peaks, then there will be diffusion of power to adjacent frequency bands through the side lobes of the filter function. Therefore, one must prewhiten, i.e., make the spectrum more nearly flat by suitable processing of the time series in the time domain, and correct afterwards in the frequency domain for the effects of prewhitening.

An example of such prewhitening by either high-pass or low-pass filtering is the processing in the time domain of the original data in the form

$$x_i' = x_i + bx_{i-1} \qquad -1 \le b \le 1 \, . \tag{25}$$

The effect in the frequency domain is to multiply by a filter function of the form

$$Y^2(w) = 1 + b^2 + 2b \cos w \tag{26}$$

so that the calculated power spectrum is

$$P(\text{calculated}) = Y^2(w)P(\text{true}) \, . \tag{27}$$

In one of the programs available, we have set $b = -0.6$. The effect of this is to reduce the low-frequency power by a factor of 0.16 relative to midband, and to increase the high-frequency power by 1.96 .

Other values of b can be used provided they are inserted into the proper locations of the program.

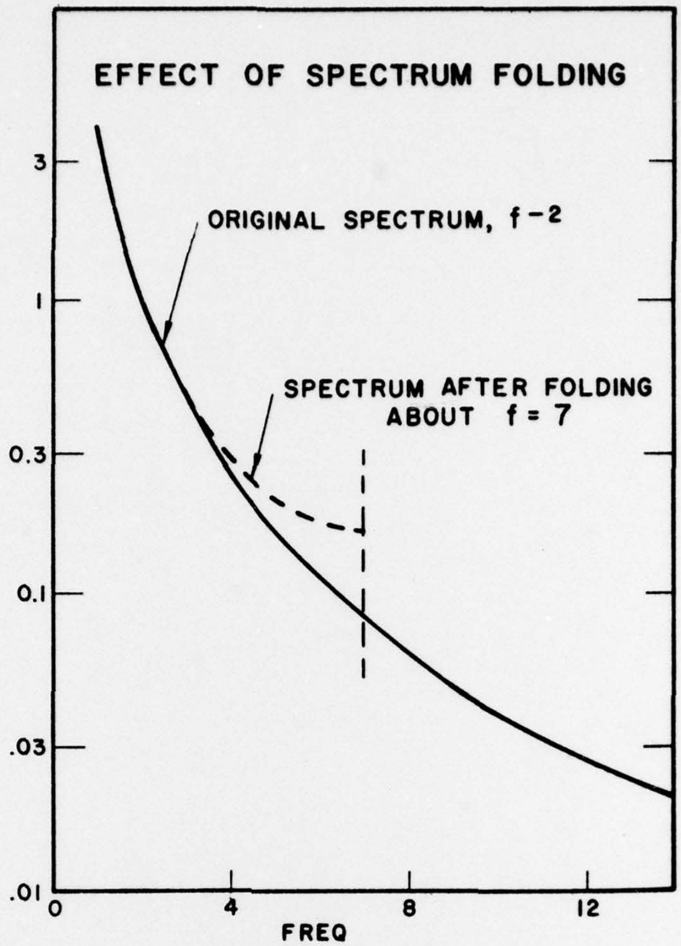
EFFECT OF SPECTRUM FOLDING
ORIGINAL SPECTRUM, f^{-2}
SPECTRUM AFTER FOLDING ABOUT $f = 7$
3
1
0.3
0.1
.03
.01
0
4
8
12
FREQ

Fig. 1

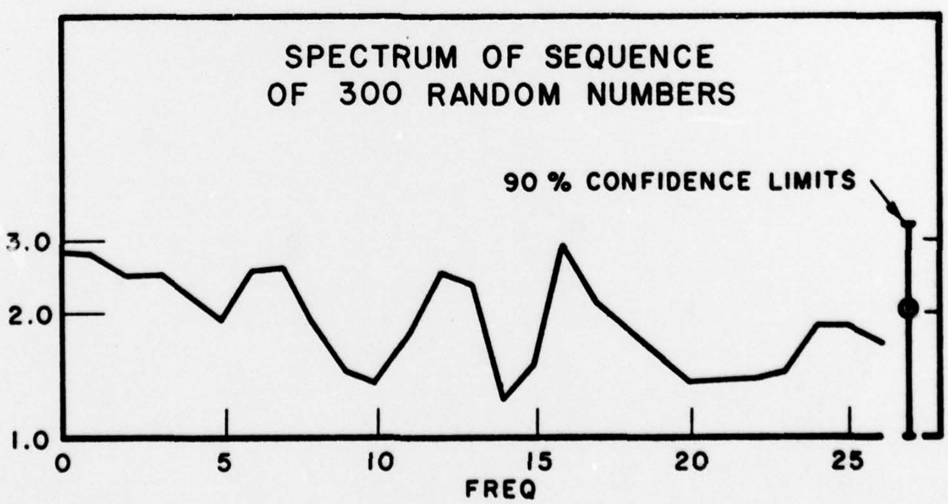
SPECTRUM OF SEQUENCE OF 300 RANDOM NUMBERS
90 % CONFIDENCE LIMITS
3.0
2.0
1.0
0
5
10
15
20
25
FREQ

Fig. 2

RESULTS

A. Noise

The spectral analysis has been applied to a sequence of random numbers. Figure 2 gives the power spectrum which is flat with fluctuations in the order of the 90 percent confidence limits (Table I). Hence, fluctuations of this order in experimental data may also be due to random effects.

B. Signal

The power spectrum of a CW signal of the form

$$y = a(1 - \cos 2\pi q/4)$$

has been calculated. The solution for the continuous case,

$$y = a \cos 2\pi f_0 t \ ,$$

may be easily calculated. The autocovariance function is

$$C(\tau) = (a^2/2) \cos 2\pi f_0 \tau \ .$$

The two-sided power spectrum is

$$P(f) = \frac{C(0)}{2} \left[\delta\left(f + f_0\right) + \delta\left(f - f_0\right) \right] \ ,$$

and the one-sided power spectrum is

$$Q(f) = C_0 \delta\left(f - f_0\right) \ .$$

The results of the Tukey analysis give for the case $f_0 = (1/4\Delta t)$ and $m = 26$,

$$2V_r = 0 \text{ for } r \neq 13$$

$$= 2mC_0 \Delta t \quad \text{for } r = 13 \ .$$

Then,

$$2U_r = 0 \quad \text{for} \quad r \neq 12,\ 13,\ 14$$

$$= \left| mC_0 \Delta t/2 \right| \quad \text{for} \quad r = 12,\ 14$$

$$= mC_0 \Delta t \quad \text{for} \quad r = 13\ .$$

The integral of the power spectrum over the frequency domain is C_0 as it should be.

C. Signal plus noise.

A small amplitude CW signal is detectable in considerable noise as the following analysis shows. Basically, this is because the power of the CW signal is concentrated at a single frequency or in a band of frequencies whereas the power in the noise is spread over the entire frequency domain as was shown in A.

Let $\varepsilon(t)$ = noise signal and $a(1 - \cos \omega t)$ = signal. The sum of the two $y(t) = \varepsilon(t) + a(1 - \cos \omega_0 t)$ is the function to be analyzed. Let the mean value of $y(t)$ be

$$\overline{y} = \overline{\varepsilon} + a$$

so

$$x(t) = y(t) - \overline{y}$$
$$= \varepsilon'(t) - a \cos \omega_0 t$$

where

$$\varepsilon'(t) = \varepsilon(t) - \overline{\varepsilon}\ .$$

Then

$$C(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} \Big[\varepsilon'(t)\varepsilon'(t+\tau) + a^2 \cos \omega_0 t \cos \omega_0 (t+\tau)$$
$$- a\varepsilon'(t+\tau) \cos \omega_0 t - a\varepsilon'(t) \cos \omega_0 (t+\tau) \Big]\, dt \qquad \text{(C.1)}$$

The first term in the integral is the autocovariance function of the noise

$$\sigma_N(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varepsilon'(t)\varepsilon'(t+\tau)dt \ .$$

Similarly, the second integral is that of the signal

$$\sigma_S(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a^2 \cos \omega_0 t \cos \omega_0 (t+\tau)dt \ .$$

The last two terms are the cross correlation of the signal and noise

$$\sigma_{SN}(\tau)$$

which, in the $\lim\limits_{T\to\infty} \sigma_{SN}(\tau) \to 0$ since the signal and noise are uncorrelated.

We can rewrite (C.1) as

$$C(\tau) = \sigma_N(\tau) + \sigma_S(\tau) + \sigma_{SN}(\tau) \tag{C.2}$$

$$\sim \sigma_N(\tau) + \sigma_S(\tau) \ .$$

The Fourier transform of $C(\tau)$ is

$$P(f) = P_N(f) + P_S(f) + P_{SN}(f) \tag{C.3}$$

$$\sim P_N(f) + P_S(f)$$

We can immediately apply (C.3) to the digital computation.

From A, the one-sided power spectrum of the noise is

$$W_r^N = \frac{\sigma_N}{\frac{1}{2\Delta t}} = 2\ \sigma_N \Delta t$$

where the letter N designates the noise spectrum.

From B, for $f_o = \frac{1}{4\Delta t}$, m = 26, the one-sided power spectrum of the signal (designated by S) is

$$W_r^{\;S} = m\,\sigma_S \Delta t \qquad \text{for} \quad r = 13$$

$$= \frac{m\,\sigma_S \Delta t}{2} \qquad \text{for} \quad r = 12,\ 14$$

$$= 0 \qquad \text{for} \quad r \neq 12,\ 13,\ 14\ .$$

The power spectrum of the signal in noise is

$$W_{r,S+N} \approx W_{r,N} + W_{r,S}\ .$$

If we let ρ designate the ratio of the spectrum level at the peak, to the background level

$$\rho = \frac{W_{13,\ S+N}}{W_{r',\ S+N}} \qquad r' \neq 12,\ 13,\ 14$$

then

$$\rho \approx 1 + \frac{W_{13,S}}{W_{r',N}}$$

$$= 1 + \frac{n\sigma_S \Delta t}{2\sigma_N \Delta t}\ ,$$

$$= 1 + \frac{n\sigma_S}{2\sigma_N}$$

therefore

$$\sigma_S = \frac{2\sigma_N}{m}\,(\rho - 1) \tag{C.4}$$

We see that a CW signal of small variance relative to the noise can be detected provided m is large enough. For a CW signal, of amplitude a,

$$\sigma_S = \frac{a^2}{2}$$

so the minimum detectable signal is

$$a = 2\sqrt{\frac{\rho - 1}{m}\sigma_N}$$

and if $\sqrt{\sigma_N} = \varepsilon$, and $\rho = 2$,

$$a = \frac{2\varepsilon}{m}. \tag{C.5}$$

PROGRAM DETAILS

Two programs are currently available for the computation of power spectra. The first computes the power spectra of short time series without preprocessing of the data. The second allows for the processing of the data by prewhitening, can handle much longer time series, but requires three steps to complete. There is also available a program for directly computing the Fourier series expansion of a time series but its use is to be discouraged as its greater frequency resolution leads to greater variability in the coefficients of the Fourier series without also yielding an estimate of the confidence to be placed in the results. Furthermore, it requires much machine time.

PROBLEM 153.001C

Direct Spectral Density Computation

Let the data be given as a set of readings,

$$Y_i \qquad i = 0,\ 1,\ 2,\ \dots\ N$$

Compute the mean value

$$\overline{Y} = \frac{1}{N} \sum_{i=0}^{N} Y_i$$

and the deviation from the mean

$$X_i = Y_i - \overline{Y}\ .$$

We compute the mean lagged product

$$C_o = \frac{1}{N} \sum_{q=0}^{N} X_q X_q \tag{16a}$$

and the normalized mean lagged products

$$C_r' = (C_r/C_o) = \frac{1}{C_o} \frac{1}{N-r} \sum_{q=0}^{N-r} X_q X_{q+r} \qquad r = 0,\ 1,\ 2,\ \dots\ m \tag{16b}$$

from which the normalized raw spectral density estimates are computed by

$$V_r' = 1 + 2 \sum_{q=1}^{m-1} C_q \cos \frac{qr\pi}{m} + C_m \cos r\pi \tag{20}$$

and the normalized refined spectral density estimates by

$$\begin{aligned} U_o' &= \frac{1}{2} \left(V_o + V_1 \right) \\ U_r' &= \frac{1}{4} V_{r-1} + \frac{1}{2} V_r + \frac{1}{4} V_{r+1} \\ U_m' &= \frac{1}{2} V_{m-1} + \frac{1}{2} V_m\ . \end{aligned} \tag{24}$$

(1) Control card information

N number of data points (1150 max.)

m maximum number of delay steps (98 max.)

Δt interval between data points time

C constants

D constants

E constants

F constants

(2) Input data

Y_i data

i index number

(3) Output

(a) C_r mean lagged product

(C_r/C_o) normalized C_r's

r

$r\Delta t$ delay time

(b) V_r' raw spectral density estimates using normalized C_r's

(c) U_r' refined spectral density estimates using normalized C_r's

r

Cr If $C = (1/2\ m\Delta t)$, then $Cr = f_r = r/2\ m\Delta t$.

DU_r (See last line below.)

EU_r If $E = 2$, then $EU_r = W_r$, one-sided power spectrum estimates.

FU_r If D and F are set equal to two times the confidence limits given in Table I, then the range within which the true value of the power density lies is given directly.

PROBLEM 153.002C

Prewhitened Spectral Density Computation

Let the data be given as a set of readings

$$Y_i \qquad i = 0, 1, 2, \ldots N$$

which is prewhitened by

$$\tilde{Y}_i = Y_i + bY_{i-1} \qquad -1 \leq b \leq 1 \ . \tag{25}$$

We then compute the mean value

$$a = \frac{1}{N} \sum_{i=1}^{N} \tilde{Y}_i \qquad i = 1, 2, \ldots N$$

and the mean lagged product

$$C_r = \left[\frac{1}{N-r} \sum_{i=1}^{i=N-r} \tilde{Y}_i \tilde{Y}_{i+r} \right] - a^2 \qquad r = 0, 1, \ldots m \ .$$

In Part II, the raw spectral density estimate is formed by

$$V_r = C_o + 2 \sum_{q=1}^{m-1} C_q \cos \frac{qr\pi}{m} + C_m (-1)^r \qquad r = 0, 1, \ldots m \tag{17}$$

and the refined spectral density estimate, corrected for prewhitening by

$$\rho_r = \frac{1}{\left(1 + b^2 + 2b \cos \frac{r\pi}{m}\right)} \left[\frac{1}{4} V_{r-1} + \frac{1}{2} V_r + \frac{1}{4} V_{r+1} \right] \qquad r = 0, 1, 2, \ldots m$$

which follows from Eq. (24), (26) and (27).

In the computation, b is set at - 0.6 , so the refined spectral density estimate is computed as

$$\rho_r = \frac{1}{\left(1.36 - 1.2 \cos \frac{r\pi}{m}\right)} \left[\frac{1}{4} V_{r-1} + \frac{1}{2} V_r + \frac{1}{4} V_{r+1} \right] \qquad r = 1, 2, \ldots m-1$$

where

$$\rho_o = \left(\frac{n}{n-m}\right)\left(\frac{1}{1.36 - 1.20 \cos \frac{\pi}{3m}}\right)\left(\frac{1}{2} V_o + \frac{1}{2} V_1\right)$$

$$\rho_m = \frac{1}{1.36 - 1.20 \cos\left(\pi - \frac{\pi}{3m}\right)} \left(\frac{1}{2} V_{m-1} + \frac{1}{2} V_m\right) .$$

However, the value of b can be altered by inserting both b^2 and 2b into the appropriate location in the program.

(1) Control card information

N, number of data points

-b, constant

(2) Input data

Y_i, i = 1, 2, 3, ... N

i, index number

(3) Output

(a) first card

Y_i,

N,

$\bar{Y}$, mean value

(b) second card, discard

(c) remaining cards

q index number, (q = k-1)

Y_q

$Y_q - \bar{Y}$ fluctuation from mean

$Y_q + bY_{q-1}$

PROGRAM 153.002 Part I

(1) Control card information

m, (fixed point)

m, (floating point)

N

(2) Input data — output of 153.00

(3) Output

(a) First card

m (fixed point)

m (floating point)

N

(b) Succeeding cards

C_r

r, (fixed point)

r

C_r, (fixed point)

PROGRAM 153.002 Part II

(1) Control card

b^2 constant, into location C

2b constant, into location D

E, F constants

(2) Input data, output of Part I

(3) Output data

(a) V_r

r

(b) $C_o U_r'$

r

Cr, if $C = \frac{1}{2m\Delta t}$, then $Cr = f_r = \frac{r}{2m\Delta t}$

DU_r, see FU_r

EU_r if $E = 2$, then $EU_r = W_r$

FU_r, if D and F are set equal to two times the confidence limits given in Table I, then the range within which the true value of the power density lies is given directly.

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